Compact Notes on Introductory Integral Calculus

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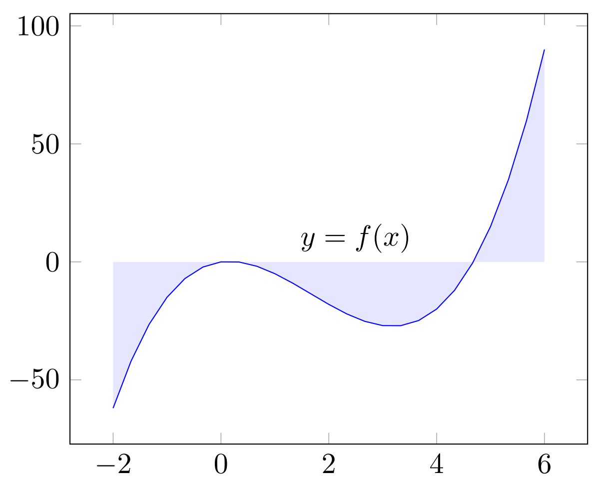
Abstract

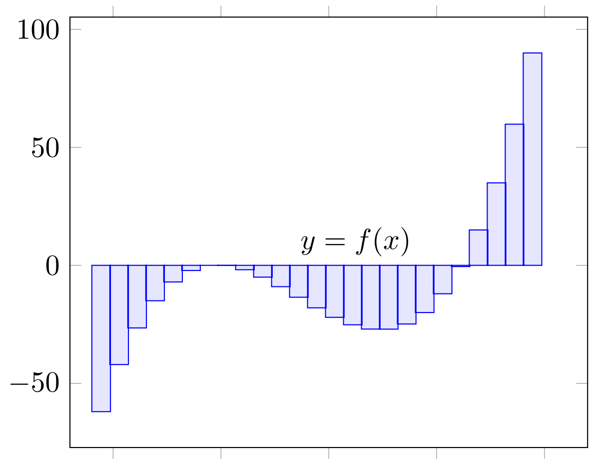
This is an introductory notes on Integral Calculus intended for 10+2 level mathematics courses. It talks about some basic results, with some geometric intuition, as well as some relevant exercises here and there. Various materials and resources are used for compilation of the notes, and the author offers thanks to them.

1 Riemann Integration

Integration is mainly used as a primary tool to calculate areas, volumes and hypervolumes. But, we shall focus on calculating areas of under a given curve.



To get the area of such a curve, we can make a partition of x-axis, then then consider some approximation of the function within each partition. This will help us to get the idea about the area under that curve, for that particular function, using the area of an approximated rectangle, within that partition. Adding such areas of all such rectangle should give us a good approximation for that area under the curve.



Note that, the approximation is better if we use more number of partitions. We should retain this idea.

What is a Partition?

A partition $\mathcal{P}$ of the closed interval $[a, b]$, is defined as a set of intervals $[x_i, x_{i+1}]$, for $i = 0, 1, 2, \ldots (n-1)$, with $x_0 = a$ and $x_n = b$, and $x_0 < x_1 < \cdots < x_n$.

Now that we have a partition, we kind of have the widths of the rectangles. However, we want the heights of those rectangles. For this, we need some tags on each partition.

What is a Tag and Tagged Partition?

A set of tags $t_1, t_2, \ldots t_n$ are chosen points in each interval, i.e. $t_i \in [x_{i-1}, x_i]$.
A tagged partition $(\mathcal{P}, \mathcal{T})$ is a pair, of which first one $\mathcal{P}$ is a partition, and the second one $\mathcal{T}$ is a set of tags.

Now that we have come to know how to choose the tagged partition, which gives us lots of rectangles, which can be used to approximate the area under the curve we want. So, we consider one such tagged partition, $(\mathcal{P}, \mathcal{T})$, and consider a rectangle of height $f(t_i)$ for the interval $[x_{i-1}, x_i]$. Then, the area of the rectangle is; $f(t_i)(x_i - x_{i-1})$. Hence, the total area of the approximating rectangles is given by;

$$S_f(\mathcal{P}, \mathcal{T}) = \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1})$$

We call this as **Riemann Sum**.

Now, note that Riemann sum is only an approximation to the area under the curve $y = f(x)$. Clearly, there are two ways to make this approximation better.

1. Increase the number of partitions.
2. Decrease the length of intervals in the partition.

However, increasing number of partition is not good enough. For example, let us assume $a = 0, b = 1$, and you consider some partition $\mathcal{P}$ where $x_1 = 0.5$. Then, you have a big rectangle that is nowhere as close as an qualifying approximation, while the other half of the domain is refined as the number of partitions gets to increase. Such a partition would not possibly lead to a good approximation of the area under curve, by means of Riemann sum.

But, the second idea seems promising and that is what is done essentially. We define a **norm** or some kind of magnitude of a partition.

Magnitude of a Partition

Magnitude of a partition $\mathcal{P}$ is defined as the maximum length among all intervals.

$$\|\mathcal{P}\| = \max_{1\leq i\leq n} (x_i - x_{i-1})$$

Now, we can formally define what is called a **Riemann Integration**.

Riemann Integrable Functions

A function $f : [a, b] \to \mathbb{R}$, is said to be Riemann Integrable, if there exists a constant $L \in \mathbb{R}$, such that; $\forall\epsilon > 0$, there exists a $\delta > 0$, such that $\forall\|\mathcal{P}\| < \delta$,

$$|S_f(\mathcal{P}, \mathcal{T}) - L| < \epsilon$$

and we say $L = \int_a^b f(x)dx$.

2 Darboux Integration

Darboux integration is another way of defining Integrable functions and calculating the area under a curve. The way to think it is by constructing two sets of rectangles, one set having area larger than the area under the curve, and one set having smaller area. Refining both simultaneously would lead us to get good approximations on both bounds, and that defines the integration.

Darboux integration can be considered as a special case of Riemann integration, however, we shall show that both are equivalent.

Upper and Lower Darboux Sums

Let us consider a partition $\mathcal{P}$. For each interval $[x_{i-1}, x_i]$, let;

$$M_i = \arg\max_{x\in[x_{i-1},x_i]} f(x)$$
$$m_i = \arg\min_{x\in[x_{i-1},x_i]} f(x)$$

Then, the Riemann sum corresponding to the tagged partition $(\mathcal{P}, \{M_i\}_{i=1}^n)$ is called **Upper Riemann Sum** or **Upper Darboux Sum**. Similarly, the Riemann sum corresponding to the tagged partition $(\mathcal{P}, \{m_i\}_{i=1}^n)$ is called **Lower Riemann Sum** or **Lower Darboux Sum**.

Now, refining this partitions, we can get the Darboux integral. But wait, we cannot do that yet. For example, consider this function $f : [0, 1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Each of the upper Darboux sum would become 1. Each of the lower Darboux sum would become 0. Hence, whatever refinement we do, these do not match up. So, there is a problem.

Exercise

Can you write it and prove that upper Darboux sum is 1 and lower Darboux sum is 0 always? What should be the area under this function?

The way to circumvent this is to define two separate quantities namely, Upper Darboux integral and Lower Darboux integral. If they match up, viola! we have the usual Riemann integration, and precisely the area under the curve.

Upper and Lower Riemann / Darboux Integral

The Upper Darboux integral is defined as the highest refinement of the partition possible, i.e. the infimum of all such partitions.

$$U_f = \inf \{S_f(\mathcal{P}, \{M_i\}_{i=1}^n) : \mathcal{P} \text{ is a partition of } [a, b]\}$$

The Lower Darboux integral is defined similarly, by taking supremum.

$$L_f = \sup \{S_f(\mathcal{P}, \{m_i\}_{i=1}^n) : \mathcal{P} \text{ is a partition of } [a, b]\}$$

Hence, if $L_f = U_f$, then we say that the function f is Darboux integrable, and $\int_a^b f(x)dx$ is the common value.

Exercise

1. Why does the set $\{S_f(\mathcal{P}, \{M_i\}_{i=1}^n) : \mathcal{P} \text{ is a partition of } [a, b]\}$ has an infimum? And why should $\{S_f(\mathcal{P}, \{m_i\}_{i=1}^n) : \mathcal{P} \text{ is a partition of } [a, b]\}$ has a supremum?

2. Show that; $L_f \leq \int_a^b f(x)dx \leq U_f$, where the middle one is value of Riemann integration.

Note that, it is clear that **Darboux integral** is a special case of **Riemann** integration, as it performs the operation of Riemann integration for a specific choice of tags, namely M_i and m_i. However, also note that, by previous exercise (2), we get;

$$L_f \leq \int_a^b f(x)dx \leq U_f$$

and since existence of Darboux integral implies $L_f = U_f$, we would have the existence of Riemann integration, as well as its value equal to the value of Darboux integral. Hence, Darboux integrals are equivalent to Riemann integrals, meaning that a function is Darboux-integrable if and only if it is Riemann-integrable, and the values of the two integrals, if they exist, are equal.

A question that should occur is which functions are actually integrable?

1. Any continuous function is integrable.

2. Any function which only finitely many discontinuities is integrable.

3. Any function which only countably infinite many discontinuities is integrable.

Use them at your convenience. The proofs of these statements require some **Measure Theory**, which is a branch of mathematics that generalizes the idea of integration. Using this, I shall show one example.

Example

We shall compute $\int_a^b x^k dx$ using definition of Riemann integration. Note that, x^k is continuous function, hence is Riemann integrable. Therefore, to compute the value of the integration, we shall use a tag of our choice, since existence of Riemann integration would ensure that any choice of tags would lead to proper evaluation of the integral.
Choose the sequence of partitions, $\mathcal{P}_n = x_0, x_1, \ldots x_n$, where $x_i = a\left(\frac{b}{a}\right)^{i/n}$. Then note that;

$$\|\mathcal{P}_n\| = \max_{1\le i\le n} a\left[\left(\frac{b}{a}\right)^{i/n} - \left(\frac{b}{a}\right)^{(i-1)/n}\right] = a\left(\frac{b}{a}\right)^{(n-1)/n}\left[\left(\frac{b}{a}\right)^{1/n} - 1\right]$$

which clearly goes to 0 as $n \to \infty$, since $\left(\frac{b}{a}\right)^{1/n} \to 1$.
Now, we choose the tags $t_i = x_i$, hence, the Riemann sum becomes,

$$\begin{aligned}
\lim_{n\to\infty} S_f(\mathcal{P}_n, \mathcal{T}) &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} a^k \left(\frac{b}{a}\right)^{ik/n} \left[a\left(\frac{b}{a}\right)^{i/n} - a\left(\frac{b}{a}\right)^{(i-1)/n}\right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} a^{k+1} \left(\frac{b}{a}\right)^{(ik+i-1)/n} \left[\left(\frac{b}{a}\right)^{1/n} - 1\right] \\
&= \lim_{n\to\infty} a^{k+1} \left(\frac{b}{a}\right)^{-1/n} \left[\left(\frac{b}{a}\right)^{1/n} - 1\right] \sum_{i=1}^{n} \left(\frac{b}{a}\right)^{i(k+1)/n} \\
&\qquad \text{After some calculation and I do not want to do that...} \\
&\qquad \text{Use that } \lim_{n\to\infty} \frac{1-(b/a)^{(1/n)}}{1-(b/a)^{((k+1)/n)}} = \frac{(-1)}{k+1} \\
&= \frac{b^{k+1} - a^{k+1}}{k+1}
\end{aligned}$$

3 Some Properties of Integral Calculus

1. **Integral is linear operator**, hence;

$$\int_a^b (\alpha f + \beta g)(x)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

for any constant α, β and any integrable functions f, g.

2. If f is an integrable function such that $f(x) \ge 0$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \ge 0$.

3. If f and g are both integrable functions, and $f(x) \le g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \ge \int_a^b g(x)dx$.

4. If f is integrable, then $|f|$ is also integrable, and $|\int_a^b f(x)dx| \geq \int_a^b |f(x)|dx$.
5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_b^c f(x)dx$, if f is integrable in above noted intervals.
6. If $f(x) > 0$ for all $x \in [a,b]$ and f is continuous in $[a,b]$, then $\int_a^b f(x)dx > 0$. Conversely, if $f(x) \geq 0$, and $\int_a^b f(x)dx = 0$, then $f(x) = 0$, for all $x \in [a,b]$, if f is continuous in $[a,b]$.
7. **Cauchy Schwartz inequality**, If f and g are two integrable functions in $[a,b]$, then;

$$\left[\int_a^b (f(x))^2 dx\right]\left[\int_a^b (g(x))^2 dx\right] \geq \left[\int_a^b f(x)g(x)dx\right]^2$$

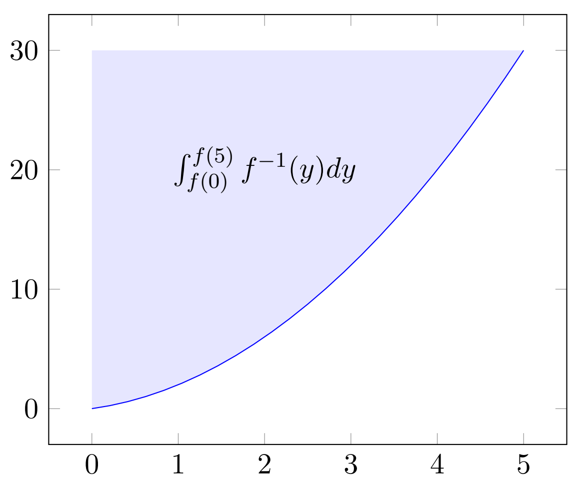
It is similar to usual Cauchy Schwartz inequality that we have learned for sequences, but the sums are replaced by integrals.

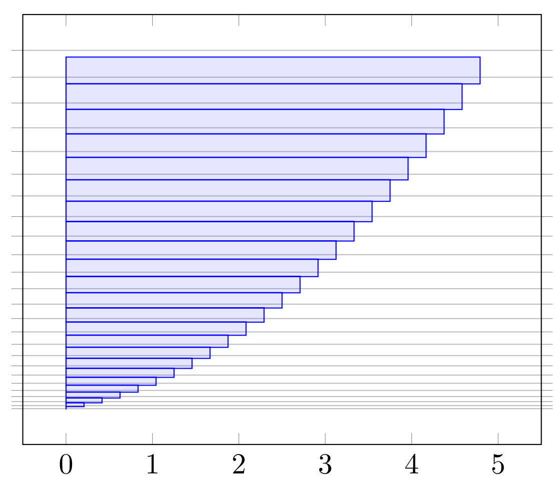
Exercise

Can you prove these properties, using definition of Riemann Integration?

4 Integral of Inverse functions

Once we know how we can use integrals to calculate the area under a curve $y = f(x)$, it basically finds the area of the region between $y = f(x)$ and the x-axis. However, for some complicated function, we may want to compute the area of the region between $y = f(x)$ and y-axis. Hence, to approximate such an area, we need to calculate the widths and heights of horizontal rectangles, rather than the vertical rectangles we were thinking before. So, we need to partition y-axis now, and the lengths of the intervals in the partition will constitute the width of the rectangle, while the height is determined by the distance between y-axis and the curve $y = f(x)$. Let us assume, we have an invertible function, and $x = f^{-1}(y)$, hence this value actually corresponds to the height of the rectangles.





Now, the question is how can we calculate that? There are two options.

1. Compute the inverse function $f^{-1}(y)$ from the form of f. This can be really problematic for a complicated function.

2. Compute the total area of the rectangle, and subtract the area of the unshaded region, i.e. the integral of f.

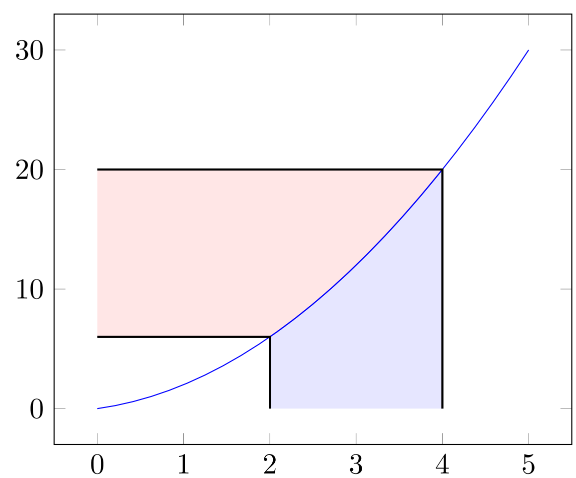
So, we consider the following results.

Computing integral of inverse functions

1. Let f be a continuous and increasing function, for which the inverse function f^{-1} exists. Then,
$$\int_{f(a)}^{f(b)} f^{-1}(y)dy = bf(b) - af(a) - \int_a^b f(x)dx$$

2. Let f be a continuous and decreasing function, for which the inverse function f^{-1} exists. Then,
$$\int_{f(b)}^{f(a)} f^{-1}(y)dy = af(a) - bf(b) + \int_a^b f(x)dx$$

The following picture should clear out the proof of the first result.



Exercise

Try to prove the second result using similar geometric proof.

5 Fundamental Theorem of Calculus

There are two broad parts of Calculus, as described in the classical theory. They are;

1. **Differential Calculus**, that deals with finding how a function changes due to infinitesimal changes of its arguments. Think of it as a generalization of subtraction.

2. **Integral Calculus**, that deals with finding the area under a given curve, or volume under a given surface, or hypervolumes in general. Think of it as a generalization of addition.

Now, we know that addition and subtraction are reverse process of each other. So, our intuition says, integration and differentiation would also be reverse process of each other. **Fundamental theorem of Calculus** establishes this fact.

First Fundamental Theorem of Calculus

Let f be integrable on $[a,b]$. For any $x \in [a,b]$, we define $F(x) = \int_a^x f(t)dt$. Then;

1. F is continuous of $[a,b]$.
2. If f is continuous at a point x_0, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Such an F is called anti-derivative of f.

Now, let us try to prove this result. However, to make the proof more understandable, we shall only consider the bounded functions.

Notes

Note that, it is possible to have unbounded functions defined on a closed and bounded interval $[0,1]$ such that, the function is integrable. For instance, the function;

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & x \in (0,1) \\ 1 & x = 0 \text{ or } 1 \end{cases}$$

is integrable, but is unbounded on $[0,1]$. Such an integral is called **Improper Integral**.

So, let us consider a bounded function f such that, $|f(t)| < M$ for all $t \in [a,b]$. Now note that,

$$|F(x) - F(y)| = |\int_a^x f(t)dt - \int_a^y f(t)dt| = |\int_y^x f(t)dt| \leq \int_y^x |f(t)|dt < M|x-y|$$

So, we can choose $\delta = \epsilon/M$ for any given $\epsilon > 0$, and then for all $|x-y| < \delta$, we have $|F(x) - F(y)| < \epsilon$, hence F is continuous.

Now, say that f is continuous at x_0. Then, for any given $\epsilon > 0$, we need to show that there exists a $\delta > 0$, such that for $|t - x_0| < \delta$,

$$|\frac{F(t) - F(x_0)}{(t - x_0)} - f(x_0)| < \epsilon$$

Now, note that, due to continuity of f at x_0, close to x_0, we have for any $|u - x_0| < \delta$;

$$\begin{aligned}
&f(x_0) - \epsilon < f(u) < f(x_0) + \epsilon \\
\Rightarrow &\int_{x_0}^t (f(x_0) - \epsilon)du < \int_{x_0}^t f(u)du < \int_{x_0}^t (f(x_0) + \epsilon)du \\
\Rightarrow &(f(x_0) - \epsilon)(t - x_0) < \int_{x_0}^t f(u)du < (f(x_0) + \epsilon)(t - x_0) \\
\Rightarrow &(f(x_0) - \epsilon) < \frac{1}{(t - x_0)} \int_{x_0}^t f(u)du < (f(x_0) + \epsilon) \\
\Rightarrow &|\frac{1}{(t - x_0)} \int_{x_0}^t f(u)du - f(x_0)| < \epsilon \\
\Rightarrow &|\frac{F(t) - F(x_0)}{(t - x_0)} - f(x_0)| < \epsilon
\end{aligned}$$

Now, there is a simple consequence of the above result, which basically shows how can we compute integral (or area under a curve), using Fundamental theorem. It is kind of a converse of First fundamental theorem.

Second Fundamental Theorem of Calculus

If f is integrable on $[a, b]$, and if there is a differentiable function F on $[a, b]$, such that $F'(x) = f(x)$ for any $x \in [a, b]$, then; $\int_a^b f(t)dt = F(b) - F(a)$.

Since, f is integrable, we can choose any partition $\mathcal{P}$ we want, and as long as $\|\mathcal{P}\| \to 0$, we should be able to retrieve the value of the integral by definition of Riemann integration. So, we consider the obvious partition, $\mathcal{P} = \{[x_i, x_{i+1}] : i = 0, 1, \ldots (n-1)\}$, where $x_i = a + (b-a)\frac{i}{(n-1)}$. Clearly, as $n \to \infty$, we have $\|\mathcal{P}\| \to 0$. Now, let us apply Lagrange's Mean Value theorem on the function F,

$$\frac{F(x_{i+1}) - F(x_i)}{(x_{i+1} - x_i)} = F'(t_i) = f(t_i)$$

where $t_i \in [x_i, x_{i+1}]$. So, we can choose the tags t_i for the interval $[x_i, x_{i+1}]$, which would lead to;

$$f(t_i)(x_{i+1} - x_i) = F(x_{i+1}) - F(x_i)$$

and summing over it would yield,

$$\begin{aligned}
\int_a^b f(t)dt &= \lim_{\|\mathcal{P}\| \to 0} \sum_{i=0}^{(n-1)} f(t_i)(x_{i+1} - x_i) \\
&= \lim_{n \to \infty} \sum_{i=0}^{(n-1)} f(t_i)(x_{i+1} - x_i) \\
&= \lim_{n \to \infty} \sum_{i=0}^{(n-1)} F(x_{i+1}) - F(x_i) \\
&= \lim_{n \to \infty} F(x_n) - F(x_0) \\
&= F(b) - F(a)
\end{aligned}$$

6 Mean Value and Some Useful Tricks of Integral Calculus

Mean Value Theorem of Integrals

If f is continuous on $[a, b]$, then there exists $x_0 \in [a, b]$ such that;

$$\int_a^b f(t)dt = f(x_0)(b-a)$$

Exercise

1. Try proving this using Lagrange's mean value theorem, as you learned in differential calculus.
2. If f and g are continuous on $[a, b]$, and $\int_a^b f(t)dt = \int_a^b g(t)dt$, then show that there exists $x_0 \in (a, b)$ such that, $f(x_0) = g(x_0)$.

Consider integrating the function $y = \sin(x^2)$. Clearly, it would have been a lot easier if we had x instead of x^2. So, this appreciation leads us to the following theorem.

Substitution Theorem

If a function $\phi(t)$ has continuous derivative $\phi'(t)$ in $[\alpha, \beta]$, and if f is continuous on an interval containing $[\alpha, \beta]$, then;

$$\int_\alpha^\beta f(\phi(t))\phi'(t)dt = \int_{\phi(\alpha)}^{\phi(\beta)} f(t)dt$$

Let $F(u) = \int_{\phi(\alpha)}^u f(t)dt$ and $H(t) = F(\phi(t))$. Then, $H'(t) = f(\phi(t))\phi'(t)$, by chain rule of differentiation. Hence,

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(t)dt = F(\phi(\beta)) = H(\beta) = \int_\alpha^\beta f(\phi(t))\phi'(t)dt$$

the last inequality following from $H(\alpha) = 0$, and second fundamental theorem of calculus.

Exercise

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for all $x \in \mathbb{R}$, $\int_0^1 f(xt)dt = 0$. Show that, f is the function which is identically equal to zero.
2. Evaluate $\lim_{n\to\infty} \left[\prod_{i=1}^n \left(1 + \frac{(2i-1)}{2n}\right)\right]^{1/2n}$.
3. If $A = \int_0^\pi \frac{\cos x}{(x+2)^2}dx$, then show that $\int_0^{\pi/2} \frac{\sin x \cos x}{(x+1)}dx = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{(\pi+2)} - A\right)$.
4. Show that, $\int_0^\pi |\frac{\sin(nx)}{x}|dx \geq \frac{2}{\pi}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$.
5. Show that, $\frac{1}{(n+1)} \leq \log(1 + \frac{1}{n}) \leq \frac{1}{n}$. (Hint is to use integrals to express logarithm).
6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let,

$$f(x) = \frac{1}{t}\int_0^t (f(x + y) - f(y))dy$$

for all $x \in \mathbb{R}$ and $t > 0$. Show that, $\frac{f(x)}{f(1)} = x$.

Now, we consider another useful theorem, the integration of product of two functions.

Integration by Parts

If f and g be integrable on $[a, b]$, and F and G denotes the anit-derivatives of them respectively, then;

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$$

The proof readily follows from the product rule of derivative. Note that;

$$\frac{d}{dx}(F(x)G(x)) = F(x)g(x) + f(x)G(x)$$

Hence, integrating both side would yield;

$$F(b)G(b) - F(a)G(a) = \int_a^b (F(x)g(x) + f(x)G(x))dx$$

this completes the proof once you take one of the term on right to the left hand side.

Exercise

1. Let $f : [0, a] \to \mathbb{R}$ be continuous. Prove that,

$$\int_0^x \left[\int_0^y f(t)dt\right] dy = \int_0^x (x - y)f(y)dy$$

2. Let p, q be positive numbers. Prove that,

$$\int_0^1 (1 - x^p)^{1/q} dx = \int_0^1 (1 - x^q)^{1/p} dx$$

3. Let $I_m = \int_0^{\pi/2} \sin^m(x)dx$. Find a recursion relation for I_m.

4. Do similar exercise for $\int_0^{\pi/2} \cos^m(x)dx$ and $\int_0^{\pi/4} \tan^m(x)dx$.

Generalized Mean Value Theorem

f, ϕ are two continuous functions on $[a, b]$ to $\mathbb{R}$. Suppose that, $\phi(x) \neq 0$ for any $x \in [a, b]$. Then, there exists a real $\xi \in [a, b]$ such that,

$$\int_a^b f(x)\phi(x)dx = f(\xi) \int_a^b \phi(x)dx$$

The proof actually follows from Cauchy's Mean Value theorem. Let, F and G denotes the anti-derivatives of $f(x)\phi(x)$ and $\phi(x)$ respectively. f and ϕ being continuous means, those anti-derivatives are differentiable. Also, $\phi(x) > 0$ or $\phi(x) < 0$ for all $x \in [a, b]$, i.e. it retains its sign on $[a, b]$ (Otherwise we have a root in between due to continuity), so $G(a) \neq G(b)$.

So, by Cauchy's MVT,

$$\frac{\int_a^b f(x)\phi(x)dx}{\int_a^b \phi(x)dx} = \frac{F(b) - F(a)}{G(b) - G(a)} = \frac{f(\xi)\phi(\xi)}{\phi(\xi)} = f(\xi)$$

We shall list some simple, yet powerful results for computation of definite integrals are as follows:

1. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$. It basically says, if you want to compute the area under the curve, you can add up rectangles from left, or from right, both would yield same result.

2. If f is an odd function, i.e. $f(-x) = -f(x)$, then $\int_{-a}^a f(x)dx = 0$. It says if we have an odd function, then the rectangles built on positive axis and on negative axis are exact mirror images of each other, with respect to both the axis.

3. If f is an even function, i.e. $f(-x) = f(x)$, then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$. It says if we have an even function, then the rectangles built on positive axis and on negative axis are exact mirror images of each other, with respect to only the x-axis.

4. For functions with $[x]$, the integral part or $\{x\}$, the fractional part, try to see the graph of the function. It helps one to calculate the area under the curve more easily, than breaking up the integral into stuffs and calculate the integral. Try calculating $\int_0^{2019}\{x\}dx$ for example.

5. For integrals of functions like, $f(x) = e^{ax}\sin(bx)$, consider the integrals of the function $g(x) = e^{ax+ibx}$, where $(i^2+1) = 0$. Here, you treat this $(a+ib)$ as real constants, then compute the integral, and finally take apart the imaginary part for the required answer. Such a computation is NOT rigorous at this level, however, you can use it to save time on MCQ type tests.

Now, we consider another trick of breaking apart integrals into partial fractions. Let us assume, we want to compute the value of the integral;

$$\int_a^b \frac{1}{(x^2+a_1^2)(x^2+a_2^2)\dots(x^2+a_k^2)}$$

We consider the form of partial fractions;

$$\frac{1}{(x^2+a_1^2)(x^2+a_2^2)\dots(x^2+a_k^2)} = \sum_{i=1}^k \frac{c_i}{(x^2+a_i^2)}$$

To compute the value of c_1 for example, we multiply both sides with $(x^2+a_1^2)$, to get;

$$\int_a^b \frac{1}{(x^2+a_2^2)\dots(x^2+a_k^2)} = c_1 + \sum_{i=2}^k \frac{c_i(x^2+a_1^2)}{(x^2+a_i^2)}$$

Now, putting, $x = a_1 i$, where $(i^2+1) = 0$, we have all partial fractions on right hand side vanished, while we get; $c_1 = \frac{1}{\prod_{i=2}^k (a_i^2 - a_1^2)}$. Hence, we finally obtain;

$$\frac{1}{(x^2+a_1^2)(x^2+a_2^2)\dots(x^2+a_k^2)} = \sum_{i=1}^k \frac{c_i}{(x^2+a_i^2)}$$

where $c_i = \frac{1}{\prod_{j\neq i}(a_j^2 - a_i^2)}$. Now, you can integrate each term separately to compute the full integral.

Another example of partial fraction could be as follows;

$$\int_a^b \frac{dx}{(x^4+2x^2\cos(\alpha)+1)}$$

Note that, $(x^4+2x^2\cos(2\alpha)+1) = (x^4+x^2(e^{ix}+e^{-ix})+1) = (x^2+e^{ix})(x^2+e^{-ix})$. Now, you can use this factorization to yield some partial fraction.

Exercise

1. Completely work out the details of the above partial fraction.

2. Let, $I_n(\alpha) = \int_0^\pi \frac{\cos(n\theta)-\cos(n\alpha)}{\cos(\theta)-\cos(\alpha)} d\theta$. Then, show that;

$$I_{n+1}(\alpha) - 2I_n(\alpha) + I_{n-1}(\alpha) = 0$$

Then compute $I_0(\alpha)$ and $I_1(\alpha)$. Using the recursion, and the initial values, show that;

$$I_n(\alpha) = n\frac{\sin(n\alpha)}{\sin(\alpha)}$$

We shall take a look into **Euler's log sine integral**. Let $I = \int_0^{\pi/2} \ln(a \sin x)dx$. Note that, $I = \int_0^{\pi/2} \ln(a\cos x)dx$. Hence,

$$\begin{aligned}
I &= \frac{1}{2}(I + I) \\
&= \frac{1}{2}\left[\int_0^{\pi/2} \ln(a\sin x)dx + \int_0^{\pi/2} \ln(a\cos x)dx\right] \\
&= \frac{1}{2}\int_0^{\pi/2} \ln(a^2 \sin x \cos x)dx \\
&= \frac{1}{2}\int_0^{\pi/2} \left[\ln(a) + \ln(1/2) + \ln(a\sin(2x))\right] dx \\
&= \frac{\pi}{4}\ln(a) + \frac{\pi}{4}\ln(1/2) + \frac{1}{2}\int_0^{\pi/2} \ln(a\sin(2x))dx
\end{aligned}$$

In the last integral, we can perform substitution, $2x = t$, and obtain;

$$I = \frac{\pi}{4}\ln(a) + \frac{\pi}{4}\ln(1/2) + \frac{I}{2}$$

from which we can solve for I.

Exercise

1. Show that, $\int_0^{\pi/2} \ln\left(\frac{\sin x}{x}\right) dx = \frac{\pi}{2}(1 - \ln(\pi))$.

2. $\int_0^\infty \frac{\ln(x^2+1)}{(x^2+1)} dx = \pi\ln(2)$. (Try trigonometric substitution)

3. $\int_0^\infty \frac{\ln(x+1/x)}{(x^2+1)} dx = \frac{\pi}{2}\ln(2)$. (Try breaking up the previous integral at 1, and then substitute $u = 1/x$ for the part with limits 1 to ∞)

4. $I_n = \int_a^b \frac{1}{(x^4+1)^n} dx$. Find a recurrence relation to express I_n in terms of $I_{(n-1)}$.

That's it! Thank you!